JOURNAL OF APPROXIMATION THEORY **10**, 237–244 (1974)

Existence and Uniqueness in Approximation by Integral Polynomials*

LE BARON O. FERGUSON†

Institut de Mathématiques, Faculté des Sciences, Université de Nancy I, 54-Nancy, France
and
Department of Mathematics, University of California, Riverside, California 92502

Communicated by Oved Shisha

In this paper we study the existence and uniqueness questions for uniform approximation over compact sets by polynomials whose coefficients are, in some sense, integers. These polynomials are the integral polynomials of the title. We also obtain some results useful in estimating the error of approximation in such cases.

Throughout, the symbol X will stand for a compact Hausdorff space and $C(X)$ (respectively, $C^R(X)$) for the set of continuous complex (respectively, real) valued functions on X. We write $\|\cdot\|_S$ for the uniform norm over S where $S \subset X$. Thus

$$\|f\|_S = \sup_{s \in S} |f(s)|.$$

We make the convention that $\|f\|_\varnothing = 0$ where $\varnothing$ is the empty set. This is convenient in the statement of Theorem 2, for example. It is also reasonable, since the quantities $|f(s)|$ always lie in $[0, \infty)$ and taking $[0, \infty)$ as the universal set leads to $\sup \varnothing = 0$ by definition. We usually write $\|\cdot\|$ in place of $\|\cdot\|_X$. If $\mathscr{F}$ is a subset of $C(X)$ (respectively, $C^R(X)$) and R a subring of the ring of complex numbers $\mathbf{C}$ (respectively, the ring of real numbers $\mathbf{R}$), we write $R[\mathscr{F}]$ for the ring of functions of the form

$$q = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} a_{i_1 \cdots i_k} f_1^{i_1} \cdots f_k^{i_k},$$

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, USAF, under Grant AFOSR-71-2030. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

† Present address: University of California.

where the a's belong to R and the f's are elements of $\mathscr{F}$. If $\mathscr{F}$ reduces to a single element f, we write $(R[\mathscr{F}])_n$ for the set of polynomials of degree at most n in f. The elements of $R[\mathscr{F}]$ are called integral polynomials. There are three cases to distinguish:

1. In the *abstract case*, X is any compact Hausdorff space, the ring R is the ring $\mathbf{Z}$ of rational integers $(\{0, \pm 1, \pm 2,...\})$, and $\mathscr{F}$ is a separating family in $C^R(X)$, that is, for any two distinct points x, y in X there exists an $f \in \mathscr{F}$ with $f(x) \neq f(y)$.

2. In the *complex case*, X is a compact subset of the complex plane. The ring R is a discrete (i.e., $0 \neq a \in R$ implies $|a| \geqslant 1$) subring of $\mathbf{C}$ (e.g., the ring of Gaussian integers $\mathbf{Z} + i\mathbf{Z}$) and $\mathscr{F}$ consists simply of the identity function z.

3. In the *real case*, X is a compact subset of the real line, the ring R is the ring of rational integers $\mathbf{Z}$, and $\mathscr{F}$ consists of the identity function x.

In all three cases we define a subset J of X as follows. Let

$$\mathscr{B} = \{q \in R[\mathscr{F}]: \| q \| < 1\}.$$

Then

$$J = \{x \in X: q(x) = 0, \text{ all } q \in \mathscr{B}\}.$$

In general, for any $G \subset C(X)$, $f \in C(X)$, and $S \subset X$ we define

$$\operatorname{dist}_S (f, G) = \inf_{g \in G} \| f - g \|_S .$$

We write $\operatorname{dist}(f, G)$ for $\operatorname{dist}_X(f, G)$. A best approximation to f from G is any $g \in G$ satisfying $\| f - g \| = \operatorname{dist}(f, G)$. The existence question is whether or not a best approximation exists for each $f \in C(X)$ $(C^R(X))$ and the uniqueness question is whether or not more than one best approximation may exist.

We first consider the existence question. As in the case of unrestricted coefficients, if we approximate by polynomials of degree at most n the existence question has an affirmative answer.

In the following we say that a subring R of $C(X)$ or $C^R(X)$ has finite rank if the linear subspace it generates has finite dimension.

THEOREM 1. *If R is a closed subring of $C(X)$ of finite rank then for every f in $C(X)$ there is a best approximation in R.*

Proof. Let $d = \operatorname{dist}(f, R)$. Then the set of best approximations to f from R is clearly

$$\bigcap_{d' > d} [(f + d'B) \cap R],$$

where B is the closed unit ball of $C(X)$. The problem thus is to show that this intersection is not void. It suffices for this to show that every $(f + d'B) \cap R$ is compact by elementary topology. Since $(f + d'B) \cap R$ is closed it suffices to show that it is contained in a compact set. Since R has finite rank the linear space V generated by it has finite dimension. V is a topological vector space with finite dimension, hence locally compact. Thus $(f + d'B) \cap V$ is compact and since it contains $(f + d'B) \cap R$ we are done. ▌

Notice that the proof, hence theorem, remains valid if we replace $C(X)$ by $C^R(X)$.

Let A be a discrete subring of the complex numbers. If X is a compact subset of $\mathbf{C}$ then $(A[z])_n$ clearly has finite rank in $C(X)$. In order to apply the theorem with $R = (A[z])_n$ then we need only show that $(A[z])_n$ is closed in $V = (\mathbf{C}[z])_n$, since the latter is finite dimensional, hence closed in $C(X)$. Suppose that $[p_k]$ is a sequence in $(A[z])_n$ converging to a polynomial p in V. Since the powers $1, z, \ldots, z^n$ form a basis for V the projections $\pi_i\colon V \to \mathbf{C}$ $(0 \leqslant i \leqslant n)$ which send each polynomial into its ith coefficient exist and are continuous on V. Thus $\pi_i(p_k) \to \pi_i(p)$ as $k \to \infty$. Since A is discrete and a subring of $\mathbf{C}$ it is closed in $\mathbf{C}$, as is well known. The $\pi_i(p_k)$ are elements of A, hence $\pi_i(p) \in A$ $(0 \leqslant i \leqslant n)$ which shows that $p \in (A[z])_n$. Since $\{p_k\}$ is any sequence in $(A[z])_n$ with a limit in V, $(A[z])_n$ is closed in V.

Notice that the same argument works in the case in which A is a discrete subring of the reals $\mathbf{R}$ (possibly $\mathbf{Z}$), X is a compact subset of $\mathbf{R}$, and $C(X)$ is replaced by $C^R(X)$.

Thus we have existence when we approximate by polynomials with integral coefficients and bounded degree. It is natural to ask if we also have existence when we approximate by polynomials with integer coefficients without a bound on the degree. Here we are approximating by a ring which is not closed in general; clearly an element f which is approximable $(\operatorname{dist}(f, \mathbf{Z}[x]) = 0)$ but not in the ring does not have a best approximation from the ring. It can also occur that a continuous function which is not approximable by the ring of all polynomials does not have a best approximation from this ring. See Andria [1, Th. 8].

We next consider the uniqueness question. Approximation by integral polynomials fails to be unique even in very simple cases. For example, consider approximation by $(\mathbf{Z}[x])_n$ on a compact subset X of the real line with $0 \in X$. Let $f \equiv 1/2$ on X. Then, since $0 \in X$, $\operatorname{dist}(f, (\mathbf{Z}[x])_n) \geqslant 1/2$ and, since the identically zero function is in $(\mathbf{Z}[x])_n$, $\operatorname{dist}(f, (\mathbf{Z}[x])_n) = 1/2$, and this for all positive integers n. The same argument shows that $\operatorname{dist}(f, \mathbf{Z}[x]) = 1/2$. Thus, in all these cases the two polynomials $p_1 \equiv 0$ and $p_2 \equiv 1$ are best approximations to f on X. There may be even more. For example, in the cases $n \geqslant 2$ in the above and for compact X satisfying

$0 \in X \subset [-1, 1]$ the polynomials $1 - x^2$ and x^2 are also best approximations.

The next theorem and its application to the real case which follows is a generalization of Andria [1, Th. 5], who proved it in the real case for X an interval $[a, b]$ with $b - a < 4$. Notice that the case $J = \varnothing$ does not cause a problem here due to the convention $\|f\|_\varnothing = 0$.

In the following theorem define J as above but with G in place of the ring $R[\mathscr{F}]$.

THEOREM 2. *Suppose*

(i) *X is a compact Hausdorff space,*

(ii) *G is a subgroup of $C(X)$ (respectively, $C^R(X)$), and*

(iii) *an element f of $C(X)$ (respectively, $C^R(X)$) is approximable* (dist$(f, G) = 0$) *if and only if there exists p in G such that $f \equiv p$ on J. Then $f \in C(x)$ (respectively, $C^R(X)$), $p \in G$ and*

$$\|f - p\|_J < \|f - p\|_X$$

imply that p is not a best approximation to f from G.

Proof. Let δ be any positive number. Set $\mu = \|f - p\|_J$ and

$$F = \{x \in X: |f(x) - p(x)| \geqslant \mu + \delta\}.$$

By the continuity of f and p, F is a closed subset of X. Also, notice from the definition of J that it is the intersection of the zeroes of the elements of G with norm strictly less than one. Since these elements are continuous, J is closed in X. Also from the definition of F we see that F and J are disjoint. Thus, by Urysohn's lemma, there exists h in $C(X)$ with $0 \leqslant h \leqslant 1$, $h(J) = \{0\}$ and $h(\mathbf{F}) = \{1\}$. Since $0 \in \mathbf{Z}$ and $h(f - p) \equiv 0$ on J there is by (iii) a $p' \in G$ satisfying

$$\| h(f - p) - p' \|_X < \delta. \tag{1}$$

Also, by the way h was constructed and the definition of F,

$$\begin{aligned} \|(f - p) - h(f - p)\|_X &= \|(f - p) - h(f - p)\|_{X \setminus F} \\ &\leqslant \| 1 - h \|_{X \setminus F} \|f - p\|_{X \setminus F} \leqslant \mu + \delta. \end{aligned} \tag{2}$$

From (1) and (2) and the triangle inequality,

$$\|(f - p) - p'\| < \mu + 2\delta.$$

Since δ was any positive number we see that p' can be chosen so as to make $\|f - (p + p')\|$ arbitrarily close to μ. Since $p + p' \in G$ we are done. ∎

Theorem 2 applies in all three of our cases, possibly under some restrictions, as follows:

In the abstract case let G be the ring $\mathbf{Z}[\mathscr{F}]$. We see from Hewitt and Zuckerman [3, Th. 6.2] that (iii) is satisfied, hence the conclusion of the theorem holds in this case.

In the complex case let A be a discrete subring of $\mathbf{C}$ with rank 2 (i.e., the real linear space generated by A is all of $\mathbf{C}$) and $G = A[z]$. If X is a subset of $\mathbf{C}$ with transfinite diameter $d(X) \geqslant 1$ then $\mathscr{B} = \{0\}$ as follows. Suppose $0 \neq q \in \mathscr{B}$. Then $\| q \| < 1$ and, dividing q by its leading coefficient we can assume that q is monic. We still have $\| q \| < 1$ since the nonzero elements of A have modulus at least unity (A is discrete). Let the degree of q be n. We have $n > 0$ since q is monic and the only monic polynomial of degree zero is $p = 1$. Define

$$M_k = \inf\{\| t \| : t \text{ monic}, \deg t = k\}.$$

Then, for every positive integer j,

$$M_{jn} \leqslant \| q^j \| = \| q \|^j;$$

hence

$$(M_{jn})^{1/jn} \leqslant \| q \|^{1/n} < 1,$$

which shows that the sequence $\{M_k^{1/k}\}$ does not have a limit greater than or equal to unity. It is well known (Hille [4, p. 226, Th. 16.1.2]) that $\{M_k^{1/k}\}$ converges to the transfinite diameter of X, a contradiction. Since $\mathscr{B} = \{0\}$ we see that, when $d(X) \geqslant 1$, $J = X$. When $J = X$, the conclusion of Theorem 2 holds vacuously. On the other hand, if $d(X) < 1$ and X is a Lavrentief subset (compact, void interior and connected complement) of the plane then by Ferguson [2, Th. 5.7 and 5.9] the hypothesis (iii) of Theorem 2 is satisfied and the conclusion holds in this case.

In the real case, if $d(X) \geqslant 1$ we see as above that $\mathscr{B} = \{0\}$ and the conclusion of Theorem 2 holds vacuously. If $d(X) < 1$, then by Ferguson [2, Th. 6.5] the hypothesis (iii) of Theorem 2 is satisfied and the conclusion of the theorem holds in this case also.

The following application of Theorem 2 is interesting in that it reduces the problem of determining the distance from an $f \in C(X)$ to the set of integral polynomials to a finite dimensional problem, in many interesting cases. We first prove the following variation on the theorem:

THEOREM 3. *Under the hypotheses of Theorem* 2, *for any f in $C(X)$ (respectively, $C^R(X)$)*

$$\operatorname{dist}_X(f, G) = \operatorname{dist}_J(f, G).$$

Proof. It is clear from definitions that $\operatorname{dist}_X(f, G) \geqslant \operatorname{dist}_J(f, G)$. To prove the reverse inequality let ϵ be any positive number. Then by definition of $\operatorname{dist}_J(f, G)$ there exists $g \in G$ with

$$\| f - g \|_J < \operatorname{dist}_J(f, G) + \epsilon/2.$$

If $\| f - g \|_X = \| f - g \|_J$ then by definition

$$\operatorname{dist}_X(f, G) < \operatorname{dist}_J(f, G) + \epsilon/2. \tag{3}$$

If $\| f - g \|_X > \| f - g \|_J$ then as in the proof of Theorem 2 we can find g' in G such that

$$\| f - g - g' \|_X \leqslant \| f - g \|_J + \epsilon/2 < \operatorname{dist}_J(f, G) + \epsilon.$$

Hence, since $g + g' \in G$ we have

$$\operatorname{dist}_X(f, G) < \operatorname{dist}_J(f, G) + \epsilon. \tag{4}$$

Since ϵ is any positive number we conclude from either (3) or (4) that

$$\operatorname{dist}_X(f, G) \leqslant \operatorname{dist}_J(f, G),$$

as was to be proved. ▮

COROLLARY 4. *Let X be a Lavrentief subset of $\mathbf{C}$ with $d(X) < 1$ and A a discrete subring of $\mathbf{C}$ with rank 2 and containing the identity. Then for any $f \in C(X)$*

(i) $\operatorname{dist}_X(f, A[z]) = \operatorname{dist}_J(f, A[z])$, *and*

(ii) *there exists $q \in A[z]$ such that*

$$\| f - q \|_J = \operatorname{dist}_J(f, A[z]).$$

Proof. We have already seen that our hypotheses here imply those of Theorem 2, hence Theorem 3, and (i) follows. Next we prove that the subset J of X is finite, as follows. Since $1 > d(X) = \lim_{k\to\infty} M_k^{1/k}$ we have $M_k < 1$ for some k. From the definition of M_k there exists a monic polynomial t with $\| t \| < 1$. By Ferguson [2, 5.3] there exists a $q \in A[z]$ with $\| q \| < 1$. Then, by definition, J is a subset of the roots of q, hence finite.

Since J is finite we can show (ii) as follows. It suffices to show that in the infemum defining $\operatorname{dist}_J(f, G)$ $(G = A[z])$ we can replace G by a finite set. Let $g_0 \in G$. Then clearly

$$\inf_{g\in G} \| f - g \|_J = \inf_{\| f-g \| \leqslant \| f-g_0 \|} \| f - g \|,$$

hence it suffices to show that

$$S_1 = \{g : \|f - g\| \leqslant \|f - g_0\|\}$$

is finite. But for any $g \in S_1$

$$\|g\| \leqslant \|f\| + \|f - g\| \leqslant \|f\| + \|f - g_0\|;$$

hence S_1 is a subset of the set

$$S_2 = \{g \in G : \|g\| \leqslant \|f\| + \|f - g_0\|\},$$

and we will be done once we show that S_2 is finite. Let $J = \{a_1, \ldots, a_n\}$. By the map

$$g \to (g(a_1), \ldots, g(a_n)) \tag{5}$$

we can identify S_2 with a subset of the closed ball of $\mathbf{C}^n$ of radius $\|f\| + \|f + g_0\|$ where $\mathbf{C}^n$ is normed by

$$|||(z_1, \ldots, z_n)||| = \max_{1 \leqslant j \leqslant n} |z_j|.$$

Next notice that G under the norm $\|\cdot\|_J$ is discrete since if $g_1, g_2 \in G$ and $\|g_1 - g_2\| < 1$ then $g_1 - g_2 \equiv 0$ on J by definition of J. Furthermore, under this same norm S_2 is a closed bounded subset of G. Since the map in (5) preserves the norm, the image in $\mathbf{C}^n$ of S_2 is discrete, bounded, and closed. It is well known that the norm $|||\cdot|||$ on $\mathbf{C}^n$ is equivalent to the usual Euclidean norm on $\mathbf{C}^n$ (as identified with $\mathbf{R}^{2n}$), hence the image of S_2 in (5) is also closed, bounded, and discrete in the usual topology on $\mathbf{C}^n$. It is therefore compact and discrete, hence finite. Since the map (5) is an injection (the elements of G must be considered here as defined on J alone, not X) the set S_2 is finite. ∎

We note in passing that we have proved something more than is contained in the statement of Corollary 4 and this might conceivably be of use in actually determining $\text{dist}_X(f, A[z])$ on a computer, for example. We see from the proof that to find $\inf_{q \in A[z]} \|f - q\|_X$ we can first take any $q_0 \in A[z]$ and then find

$$\inf_{\substack{q \in A[z] \\ \|q\| \leqslant \|f\| + \|f - q_0\|}} \|f - q\|_J,$$

which entails only a finite number of calculations.

The last result also holds in the real case. The comment of the last paragraph applies here as well, with $A[z]$ replaced by $\mathbf{Z}[x]$.

COROLLARY 5. *Let X be a compact subset of R with $d(X) < 1$. Then for any f in $C^R(X)$ we have*

640/10/3-4

(i) $\text{dist}_X(f, \mathbf{Z}[x]) = \text{dist}_J(f, \mathbf{Z}[x])$, *and*

(ii) *there exists* q *in* $\mathbf{Z}[x]$ *such that*

$$\| f - q \|_J = \text{dist}_J(f, \mathbf{Z}[x]).$$

The proof of this corollary parallels that of Corollary 4 with minor changes. The proof that J is finite in this case follows from the corresponding argument in the proof of Corollary 4 plus Proposition 6.2 in Ferguson [2, p. 64].

REFERENCES

1. G. ANDRIA, Approximation of continuous functions by polynomials with integral coefficients, *J. Approximation Theory* **4** (1971), 357–362.
2. L. B. O. FERGUSON, Uniform approximation by polynomials with integral coefficients I, *Pacific J. Math.* **27** (1968), 53–69.
3. E. HEWITT AND H. S. ZUCKERMAN, Approximation by polynomials with integral coefficients, a reformulation of the Stone-Weierstrass theorem, *Duke Math. J.* **26** (1959), 305–324.
4. E. HILLE, "Analytic Function Theory," Vol. II, Ginn, Waltham, Mass., 1962.